

A LEFSCHETZ (1,1)-THEOREM FOR NORMAL PROJECTIVE COMPLEX VARIETIES

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1. INTRODUCTION

Let X be a projective variety over \mathbb{C} . Let X_{an} be the analytic space associated to X . Let $c_1 : Pic(X) \rightarrow H^2(X_{an}, \mathbb{Z})$ be the map which associates to a line bundle (or equivalently a Cartier divisor) on X its cohomology class. We may identify the Néron-Severi group $NS(X)$ with the image of $Pic(X)$ in $H^2(X_{an}, \mathbb{Z})$ under the above map.

If X is smooth, then by the Hodge decomposition theorem, we know that

$$H^2(X_{an}, \mathbb{C}) = H^{2,0}(X_{an}) \oplus H^{1,1}(X_{an}) \oplus H^{0,2}(X_{an}).$$

Let $F^1 H^2(X_{an}, \mathbb{C}) = H^{2,0}(X_{an}) \oplus H^{1,1}(X_{an})$. The *Lefschetz theorem on (1,1) classes* ([GH], [L]) states that if X is a smooth, projective variety, then

$$NS(X) = \{\alpha \in H^2(X_{an}, \mathbb{Z}) \mid \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}.$$

If X is an arbitrary singular variety then by [D], Theorem 8.2.2 the cohomology groups of X with \mathbb{Z} -coefficients carry mixed Hodge structures. Hence it makes sense to talk of $F^1 H^2(X_{an}, \mathbb{C})$ for such a variety X . Spencer Bloch, in a letter to Jamsen [J, appendix A], asks whether the “obvious” extension of the Lefschetz (1,1) theorem is true for singular projective varieties, i.e., is it true that

$$NS(X) = \{\alpha \in H^2(X_{an}, \mathbb{Z}) \mid \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}?$$

Barbieri-Viale and Srinivas [BS1] gave a counterexample to this question. Let X be a surface defined by the homogenous equation $w(x^3 - y^2z) + f(x, y, z) = 0$ in $\mathbb{P}_{\mathbb{C}}^3$, where x, y, z, w are homogenous coordinates in $\mathbb{P}_{\mathbb{C}}^3$ and f is a “general” homogenous polynomial over \mathbb{C} of degree 4. They showed that for such an X ,

$$NS(X) \subsetneq \{\alpha \in H^2(X_{an}, \mathbb{Z}) \mid \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}.$$

In the same paper [BS1] the authors ask the following question. Let X be a complete variety over \mathbb{C} . Let $H^1(X, \mathcal{H}_X^1)$ be the subgroup of $H^2(X_{an}, \mathbb{Z})$ consisting of Zariski-locally trivial cohomology classes, i.e., $\eta \in H^2(X_{an}, \mathbb{Z})$ lies in $H^1(X, \mathcal{H}_X^1)$ if and only if there exists a finite open cover $\{U_i\}$ of X by Zariski open sets such that $\eta \mapsto 0$ under the restriction maps $H^2(X_{an}, \mathbb{Z}) \rightarrow H^2((U_i)_{an}, \mathbb{Z})$ for all i . Is

$$NS(X) = \{\alpha \in H^1(X, \mathcal{H}_X^1) \mid \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}?$$

We remark that for a smooth, projective variety X , if a cohomology class $\eta \in H^2(X_{an}, \mathbb{Z})$ is zero when restricted to a nonempty Zariski open set $U \subset X$, then η is the class of a divisor. So $H^1(X, \mathcal{H}_X^1) = NS(X)$ and the above question has a positive answer for a smooth, projective variety X . For any projective

variety X , there is an inclusion $NS(X) \subset H^1(X, \mathcal{H}_X^1)$; Barbieri-Viale and Srinivas also give an example in [BS1] of a singular variety for which this inclusion is strict.

In general, for any projective variety X over \mathbb{C} ,

$$NS(X) \subset \{\alpha \in H^1(X, \mathcal{H}_X^1) | \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}.$$

This follows from the inclusion $NS(X) \subset H^1(X, \mathcal{H}_X^1)$, combined with

$$Ker(H^2(X_{an}, \mathbb{C}) \rightarrow H^2(X_{an}, \mathcal{O}_{X_{an}})) \subset F^1 H^2(X_{an}, \mathbb{C}),$$

which is a consequence of results of Du Bois [DB] (and is also implicit in [D]). When X is normal, we prove the reverse inclusion, thereby answering the question in the affirmative, for the normal case. The statement of our Main Theorem is

Theorem 1.1. *Let X be a normal, projective variety over \mathbb{C} . Then*

$$NS(X) = \{\alpha \in H^1(X, \mathcal{H}_X^1) | \alpha_{\mathbb{C}} \in F^1 H^2(X_{an}, \mathbb{C})\}.$$

We also describe a counterexample, of a non-normal irreducible projective 3-fold with smooth normalization (isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) for which the question has a *negative* answer. However, it seems likely that the conclusion of the Theorem holds for any *semi-normal* projective variety X over \mathbb{C} .

Our proof of Theorem 1.1 is in two steps: first we show that $H^1(X, \mathcal{H}_X^1) \subset H^2(X_{an}, \mathbb{Z})$ is a sub-MHS of level 1; hence by [D], it determines a 1-motive, which we show to be an extension of a direct sum of Tate structures $\mathbb{Z}(-1)$ by that of $(H^1 \text{ of})$ an abelian variety. In the second part of the proof, we give a direct construction of a certain 1-motive, using the Zariski topology on X , and show that it is isogenous to the earlier one. The Theorem will be an immediate corollary.

In a future work, we hope to use the second, algebraically defined 1-motive to also obtain the analogue of the Tate conjecture in our situation, which would similarly characterize the \mathbb{Z}_{ℓ} -span of the classes of Cartier divisors in $H_{\text{et}}^2(X_{\bar{K}}, \mathbb{Z}_{\ell}(1))$, for a normal projective variety X over a number field K .

In another direction, our result suggests a question analogous to the Hodge conjecture. Let X be a normal projective variety over \mathbb{C} , and $\alpha : X_{an} \rightarrow X$ be the obvious continuous map, leading to a Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q \alpha_* \mathbb{Q}_{X_{an}}) \implies H^{p+q}(X_{an}, \mathbb{Q}),$$

with an induced decreasing *Leray filtration* $\{L^p H^n(X_{an}, \mathbb{Q})\}_{p \geq 0}$ on each cohomology group $H^n(X_{an}, \mathbb{Q})$. Let

$$\text{Hg}^p(X) = L^p H^{2p}(X_{an}, \mathbb{Q}) \cap F^p H^{2p}(X_{an}, \mathbb{C}).$$

Is $\text{Hg}^p(X)$ the image of the p -th Chern class map $c_p : K_0(X) \otimes \mathbb{Q} \rightarrow H^{2p}(X, \mathbb{Q})$? Note that this does not hold without some hypothesis like (at least) normality; for example, Bloch's letter to Jannsen [J, Appendix A] gives a counterexample. On the other hand, results of Collino [Co] imply it when X has a unique singular point. Note also that, unlike the standard Hodge conjecture, the positive answer (our Theorem above) for divisors does not automatically imply a positive answer for the case of 1-cycles, since we do not have Poincaré duality.

This work formed part of the first author's Ph.D. thesis, written at the Tata Institute of Fundamental Research, Mumbai, submitted to the Mumbai University in March, 1997.

2. SOME PRELIMINARIES

2.1. Constructible sheaves. We will need below some technical results on constructible sheaves on complex algebraic varieties. We begin by recalling the appropriate definitions, the first from [V] and the second from [BS2].

Definition 2.1. Let X be a complex algebraic variety. We say that a sheaf \mathcal{F} of abelian groups on the analytic space X_{an} is (algebraically) \mathbb{Z} -constructible if there is a finite decomposition $X = \cup_{i \in I} X_i$, where each X_i is irreducible and Zariski closed in X , such that if $U_i = X_i - \cup_{X_j \subsetneq X_i} X_j$, then each U_i is non-singular, X is the disjoint union of the U_i , and $\mathcal{F}|_{(U_i)_{an}}$ is a locally constant sheaf whose fibre is a finitely generated group. We call any such collection of subsets $\{X_i\}_{i \in I}$ an *admissible family* of subsets for \mathcal{F} .

Definition 2.2. A sheaf \mathcal{G} on a scheme X (over an algebraically closed field k , say) is said to be \mathbb{Z} -constructible for the Zariski topology if we can express X as a finite union $X = \cup X_i$, where $X_i \subset X$ are Zariski closed, such that if $U_i = X_i - \cup_{X_j \subsetneq X_i} X_j$, then each U_i is non-singular, X is the disjoint union of the U_i , and $\mathcal{G}|_{U_i}$ is a *constant sheaf* associated to a finitely generated abelian group. We call any such collection of subsets $\{X_i\}_{i \in I}$ an *admissible family* for \mathcal{G} .

Remark 2.3. We note that in the cited works, it is not required that the “open strata” U_i are non-singular, but this may clearly be assumed as well without loss of generality, by refining any given stratification which has all the remaining properties.

Note that if $\{X_i\}_{i \in I}$ is an admissible family of subsets for a \mathbb{Z} -constructible sheaf in either of the senses above, then there is a natural partial order on the index set I given by $j \leq i \iff X_j \subset X_i$. Then we clearly have $U_i = X_i - \cup_{j < i} X_j$. Note that $U_i = X_i$ precisely when i is a minimal element of I with respect to the partial order.

We recall the following basic result from [V], which is made use of below.

Theorem 2.4. *If $f : Y \rightarrow X$ is a morphism of \mathbb{C} -varieties and \mathcal{F} is a \mathbb{Z} -constructible sheaf on Y_{an} , then $R^i f_* \mathcal{F}$ is a \mathbb{Z} -constructible sheaf on X_{an} .*

We also need a certain general sheaf-theoretic result, which is presumably well-known, but for which we do not know a reference. Let X be a topological space, $\{U_i\}_{i \in I}$ any finite collection of locally closed subsets of X which stratify X (i.e., X is the disjoint union of the U_i , and for each i , the closure $X_i := \bar{U}_i$ is again a union of some U_j). Let \leq denote the obvious partial order on I , given by $i \leq j \iff X_i \subset X_j$.

Let $f_i : U_i \rightarrow X$ be the inclusion. If \mathcal{F} is a sheaf of abelian groups on X , and $\{i_0 \leq i_1 \leq \dots \leq i_p\}$ is a p -chain in I , let

$$\mathcal{F}_{i_0 \dots i_p} := (f_{i_0})_* f_{i_0}^{-1} (f_{i_1})_* f_{i_1}^{-1} \dots (f_{i_p})_* f_{i_p}^{-1} \mathcal{F}.$$

Note that the sheaves $\mathcal{F}_{i_0 \dots i_p}$ define a cosimplicial sheaf on the simplicial space $X \times N(I)$, where $N(I)$ denotes the nerve of I (regarded as a discrete simplicial space). The augmentation $X \times N(I) \rightarrow X$ gives rise to a complex of sheaves on X

$$0 \rightarrow \mathcal{F} \rightarrow \bigoplus_{i \in I} \mathcal{F}_i \rightarrow \bigoplus_{\{i_0 \leq i_1\} \in N_1(I)} \mathcal{F}_{i_0 i_1} \rightarrow \dots \rightarrow \bigoplus_{\{i_0 \leq \dots \leq i_p\} \in N_p(I)} \mathcal{F}_{i_0 \dots i_p} \rightarrow \dots \quad (*)$$

Lemma 2.5. *The above complex $(*)$ is a resolution of \mathcal{F} .*

Proof. If $x \in U_i$, then taking the stalks at x , we have an associated cosimplicial abelian group $(\mathcal{F}_{i_0 \dots i_p})_x$, and a corresponding augmented complex. Clearly $(\mathcal{F}_{i_0 \dots i_p})_x = 0$ unless $i \leq i_0$. Since the partially ordered subset $(I \geq i) = \{j \in I \mid i \leq j\}$ has a minimal element, one sees easily that the stalk complex at x is contractible (note that if $x \in U_i$, and $\sigma = \{i_0 \leq \dots \leq i_p\}$ is a p -simplex in the nerve of $(I \geq i)$, the stalks at x of $\mathcal{F}_{i_0 \dots i_p}$ and $\mathcal{F}_{i i_0 \dots i_p}$ are naturally isomorphic, where $\{i \leq i_0 \leq \dots \leq i_p\}$ is the cone over σ with vertex i). \square

Remark 2.6. In case \mathcal{F} is \mathbb{Z} -constructible for the Zariski topology on a scheme X , and $\{X_i\}$ is an admissible family for \mathcal{F} , such that $\mathcal{F}|_{U_i}$ is the constant sheaf associated to A_i , then $\mathcal{F}_{i_0 \dots i_p}$ is just the constant sheaf $(A_{i_p})_{X_{i_0}}$. In particular, for a \mathbb{Z} -constructible sheaf in the Zariski topology, we obtain a *flasque resolution*.

The key technical result of this section is the following.

Lemma 2.7. *Let $\mathcal{A} = A_{X_{an}}$ be a constant sheaf on a complex algebraic variety X , and let \mathcal{G} be a \mathbb{Z} -constructible sheaf on X_{an} . Let $f : \mathcal{A} \rightarrow \mathcal{G}$ be a sheaf homomorphism, and take $\mathcal{F} = \text{image } f$. Let $a : X_{an} \rightarrow X$ be the natural continuous map from the analytic space X_{an} to X , which is the identity on points. Then we have the following.*

- (i) $a_* \mathcal{F}$ is a constructible sheaf on X for the Zariski topology.
- (ii) The natural map $a^{-1} a_* \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism, and the natural map $a_* \mathcal{A} \rightarrow a_* \mathcal{F}$ is surjective, i.e., $a_* \mathcal{F}$ is a quotient of the constant sheaf on X associated to the abelian group A .
- (iii) Let $\{X_i\}_{i \in I}$ be an admissible family of subsets for \mathcal{G} . Then it is also admissible for \mathcal{F} , and for $a_* \mathcal{F}$. There is an exact sequence

$$0 \rightarrow H^0(X_{an}, \mathcal{F}) \rightarrow \bigoplus_{i \in I} H^0((U_i)_{an}, \mathcal{F}|_{(U_i)_{an}}) \rightarrow \bigoplus_{\substack{i \leq j \\ i, j \in I}} H^0((U_j)_{an}, \mathcal{F}|_{(U_j)_{an}})$$

Proof. We first claim that if $U \subset X$ is an irreducible (Zariski) locally closed subset such that $\mathcal{G}|_{U_{an}}$ is locally free, then $\mathcal{F}|_{U_{an}}$ is a constant sheaf associated to a finitely generated abelian group, which is a quotient of A . Indeed, $\mathcal{G}|_{U_{an}}$ corresponds to a representation of the fundamental group of U_{an} (with respect to

any convenient base point), while $A_{U_{an}}$ corresponds to the trivial representation. The sheaf map $f|_{U_{an}}$ is then a morphism of local systems, whose image $\mathcal{F}|_{U_{an}}$ is clearly a trivial (*i.e.*, constant) local subsystem of $\mathcal{G}|_{U_{an}}$.

Now let $\{X_i\}_{i \in I}$ be admissible for \mathcal{G} . As observed above, $\mathcal{F}|_{(U_i)_{an}}$ is constant for each i , and so $\{X_i\}_{i \in I}$ is also admissible for \mathcal{F} . From lemma 2 of [BS2], it follows that $a_*\mathcal{F}$ is \mathbb{Z} -constructible for the Zariski topology.

From the beginning of the exact sequence (*) of lemma 2.5 (for \mathcal{F} on X_{an}) we have inclusions

$$\mathcal{F} \hookrightarrow \bigoplus_{i \in I} \mathcal{F}_i, \quad a_*\mathcal{F} \hookrightarrow \bigoplus_{i \in I} a_*\mathcal{F}_i, \quad a^{-1}a_*\mathcal{F} \hookrightarrow \bigoplus_{i \in I} a^{-1}a_*\mathcal{F}_i.$$

We see at once from the definitions that $a_*\mathcal{F}_i$ is (the direct image on X of) a constant sheaf on X_i , for each i , and the natural sheaf map $a^{-1}a_*\mathcal{F}_i \rightarrow \mathcal{F}_i$ is injective. Since \mathcal{A} is a constant sheaf, we also have that $a^{-1}a_*\mathcal{A} \rightarrow \mathcal{A}$ is an isomorphism. Now from the commutative diagram

$$\begin{array}{ccccc} a^{-1}a_*\mathcal{A} & \longrightarrow & a^{-1}a_*\mathcal{F} & \hookrightarrow & \bigoplus_{i \in I} a^{-1}a_*\mathcal{F}_i \\ \cong \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{F} & \hookrightarrow & \bigoplus_{i \in I} \mathcal{F}_i \end{array}$$

we deduce that $a^{-1}a_*\mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism, and that the natural map $a^{-1}a_*\mathcal{A} \rightarrow a^{-1}a_*\mathcal{F}$ is surjective. This implies that $a_*\mathcal{A} \rightarrow a_*\mathcal{F}$ is surjective as well, and that $\{X_i\}$ is admissible for $a_*\mathcal{F}$. The exact sequence in (iii) of the lemma is obtained from the resolution of lemma 2.5 for $a_*\mathcal{F}$. \square

2.2. A homological lemma. We prove here an abstract homological lemma (lemma 2.8) which is a variant of a lemma in [PS], which we will need later. The lemma is formulated and proved with abelian groups, but a similar argument yields it in an arbitrary abelian category. Suppose we have the following 9-diagram, in the category of complexes of abelian groups, with exact rows and columns.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{11}^\bullet & \longrightarrow & C_{12}^\bullet & \longrightarrow & C_{13}^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{21}^\bullet & \longrightarrow & C_{22}^\bullet & \longrightarrow & C_{23}^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{31}^\bullet & \longrightarrow & C_{32}^\bullet & \longrightarrow & C_{33}^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Applying the cohomology functor we get an infinite double sequence with exact rows and columns as shown below:

$$\begin{array}{ccccccccc}
H^{i-2}(C_{22}^\bullet) & \longrightarrow & H^{i-2}(C_{23}^\bullet) & \longrightarrow & H^{i-1}(C_{21}^\bullet) & \longrightarrow & H^{i-1}(C_{22}^\bullet) & \longrightarrow & H^{i-1}(C_{23}^\bullet) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^{i-2}(C_{32}^\bullet) & \longrightarrow & H^{i-2}(C_{33}^\bullet) & \longrightarrow & H^{i-1}(C_{31}^\bullet) & \longrightarrow & H^{i-1}(C_{32}^\bullet) & \longrightarrow & H^{i-1}(C_{33}^\bullet) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^{i-1}(C_{12}^\bullet) & \longrightarrow & H^{i-1}(C_{13}^\bullet) & \longrightarrow & H^i(C_{11}^\bullet) & \longrightarrow & H^i(C_{12}^\bullet) & \longrightarrow & H^i(C_{13}^\bullet) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^{i-1}(C_{22}^\bullet) & \longrightarrow & H^{i-1}(C_{23}^\bullet) & \longrightarrow & H^i(C_{21}^\bullet) & \longrightarrow & H^i(C_{22}^\bullet) & \longrightarrow & H^i(C_{23}^\bullet) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^{i-1}(C_{32}^\bullet) & \longrightarrow & H^{i-1}(C_{33}^\bullet) & \longrightarrow & H^i(C_{31}^\bullet) & \longrightarrow & H^i(C_{32}^\bullet) & \longrightarrow & H^i(C_{33}^\bullet) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

Suppose now that we have an element $\alpha \in H^i(C_{rs}^\bullet)$ (say for example $\alpha \in H^i(C_{22}^\bullet)$ in the above diagram) such that $\alpha \mapsto 0$ under both the maps with domain $H^i(C_{rs}^\bullet)$. We can then do a diagram chase in the above cohomology diagram in the following way. Suppose $\alpha \in H^i(C_{22}^\bullet)$; arbitrarily choose lifts $\beta_1 \in H^i(C_{21}^\bullet)$ and $\beta_2 \in H^i(C_{12}^\bullet)$ lifting α . Let $\beta_1 \mapsto \gamma_1 \in H^i(C_{31}^\bullet)$ and let $\beta_2 \mapsto \gamma_2 \in H^i(C_{13}^\bullet)$. Then since $\gamma_1 \mapsto 0 \in H^i(C_{32}^\bullet)$ and $\gamma_2 \mapsto 0 \in H^i(C_{23}^\bullet)$, there exist δ_1 and δ_2 , both in $H^{i-1}(C_{33}^\bullet)$, lifting γ_1 and γ_2 respectively.

We can do a similar diagram chase beginning with an element $\alpha \in H^i(C_{rs}^\bullet)$, for arbitrary i, r, s , and end up with two elements δ_1, δ_2 in the same group $H^j(C_{r+1\ s+1}^\bullet)$, where we read the subscripts modulo 3, and j is either $i-1$, i or $i+1$, depending on (r, s) (we end up at the two places in the diagram which have the same entry, and are each 1 ‘knight’s move’ away from the starting point).

Let $\bar{H}^j(C_{r+1\ s+1}^\bullet)$ denote the quotient of $H^j(C_{r+1\ s+1}^\bullet)$ by the subgroup generated by the images of the two maps in the large commutative cohomology diagram with range $H^j(C_{r+1\ s+1}^\bullet)$. For example,

$$H^{\bar{i}-1}(C_{3,3}^\bullet) = \frac{H^{i-1}(C_{33}^\bullet)}{\text{image } H^{i-1}(C_{23}^\bullet) + \text{image } H^{i-1}(C_{32}^\bullet)}.$$

Lemma 2.8. *With the notation as above, we have*

$$(\delta_1 - \delta_2) \mapsto 0 \in \bar{H}^j(C_{r+1\ s+1}^\bullet).$$

Proof. We first note that, by an argument with mapping cones and cylinders (rotating the distinguished triangles in the 9-diagram), we may assume that $\alpha \in H^i(C_{22}^\bullet)$ without loss of generality. For such an α the analogous result for the cohomology diagram arising from a 9-diagram in the category of sheaves has been proved by Parimala and Srinivas [PS, Sec 3]. The proof of this lemma is

entirely analogous: regarding the given 9-diagram as a (bounded) double complex of complexes, one considers the total complex, which is a 5-term exact sequence of complexes, say

$$0 \rightarrow \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \mathcal{C}_3 \rightarrow \mathcal{C}_4 \rightarrow 0.$$

Regarding this again as a double complex, there is a spectral sequence

$$E_1^{r,s} = H^s(\mathcal{C}_r) \implies H^{r+s}(Tot(\mathcal{C}_\bullet))$$

(the limit is in fact 0). Then the conclusion of the lemma is interpreted as giving two (equivalent) ways of computing the differential $E_2^{2,i} \rightarrow E_2^{4,i-1}$. \square

Remark 2.9. An analogue of lemma 2.8 can be formulated for a 9-diagram in the derived category of abelian groups

$$\begin{array}{ccccccc} C_{11} & \longrightarrow & C_{12} & \longrightarrow & C_{13} & \longrightarrow & C_{11}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_{21} & \longrightarrow & C_{22} & \longrightarrow & C_{23} & \longrightarrow & C_{21}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_{31} & \longrightarrow & C_{32} & \longrightarrow & C_{33} & \longrightarrow & C_{31}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_{11}[1] & \longrightarrow & C_{12}[1] & \longrightarrow & C_{13}[1] & \longrightarrow & C_{11}[2] \end{array}$$

where the rows and columns are distinguished triangles, and where the cohomology diagram considered earlier is replaced by the diagram obtained by applying any abelian group valued cohomological functor (of course a still more general formulation is also possible). This is *false*; O. Gabber has kindly shown us a counterexample.

Remark 2.10. A version of the above lemma 2.8 also appears in a letter from U. Jannsen to B. Gross.

3. A SHORT EXACT SEQUENCE OF MIXED HODGE STRUCTURES

In this section we make an analysis of the mixed Hodge structure on

$$H^1(X, \mathcal{H}_X^1) = \text{subgroup of Zariski locally trivial elements in } H^2(X_{an}, \mathbb{Z}).$$

Our goal is to describe it as an extension of a direct sum of Tate Hodge structures $\mathbb{Z}(-1)$ by a polarizable pure Hodge structure of weight 1.

Let X be our given normal projective variety over \mathbb{C} . Let Y be a resolution of singularities of X , and let Y_{an} be the associated analytic space of Y . We have the following commutative diagram

$$\begin{array}{ccc} Y_{an} & \xrightarrow{a^Y} & Y \\ \downarrow \pi^{an} & & \downarrow \pi \\ X_{an} & \xrightarrow{a^X} & X \end{array}$$

The Leray spectral sequence for the constant sheaf $\mathbb{Z} = \mathbb{Z}_{Y_{an}}$ and the map $\pi^{an} : Y_{an} \rightarrow X_{an}$ leads to an exact sequence:

$$(3.1) \quad 0 \rightarrow H^1(X_{an}, \mathbb{Z}) \rightarrow H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z}) \rightarrow H^2(X_{an}, \mathbb{Z}) \rightarrow H^2(Y_{an}, \mathbb{Z})$$

Note that since X is normal, we have $\pi_*^{an}\mathbb{Z} \cong \mathbb{Z}$.

Define a new sheaf $\mathcal{F}_{\mathbb{Z}}$ on X_{an} by

$$(3.2) \quad \mathcal{F}_{\mathbb{Z}} = \text{image}(H^1(Y_{an}, \mathbb{Z})_{X_{an}} \rightarrow R^1\pi_*^{an}\mathbb{Z}).$$

Here by $H^1(Y_{an}, \mathbb{Z})_{X_{an}}$ we mean the constant sheaf on X_{an} associated to the group $H^1(Y_{an}, \mathbb{Z})$, and the map on sheaves is induced at the level of presheaves by the restriction map on cohomology $H^1(Y_{an}, \mathbb{Z}) \rightarrow H^1((\pi^{an})^{-1}(U_{an}), \mathbb{Z})$ where $U_{an} \subset X_{an}$ is open. By taking global sections we have the following commutative diagram,

$$\begin{array}{ccc} H^1(Y_{an}, \mathbb{Z}) & \longrightarrow & H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \\ & \searrow & \downarrow \\ & & H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z}) \end{array}$$

Hence we have an inclusion,

$$0 \rightarrow \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \rightarrow \frac{H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} = \text{Ker}(H^2(X_{an}, \mathbb{Z}) \rightarrow H^2(Y_{an}, \mathbb{Z}))$$

where the last equality is due to the above exact sequence (3.1) of low degree terms of the Leray spectral sequence.

Note that $\mathcal{F}_{\mathbb{Z}}$ satisfies the hypotheses of lemma 2.7, with $A = H^1(Y_{an}, \mathbb{Z})$ and $\mathcal{G} = R^1\pi_*^{an}\mathbb{Z}$ (the latter is algebraically \mathbb{Z} -constructible by theorem 2.4). Hence the following properties hold.

- (i) $\mathcal{F}_{\mathbb{Z}}$ is algebraically \mathbb{Z} -constructible.
- (ii) $a_*^X \mathcal{F}_{\mathbb{Z}} := \mathcal{G}_{\mathbb{Z}}$ is \mathbb{Z} -constructible for the Zariski topology, and $(a^X)^* \mathcal{G}_{\mathbb{Z}} \cong \mathcal{F}_{\mathbb{Z}}$.
- (iii) The natural sheaf map

$$(3.3) \quad H^1(Y_{an}, \mathbb{Z})_X \rightarrow a_*^X \mathcal{F}_{\mathbb{Z}}$$

is surjective.

Lemma 3.1. $\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \subset H^1(X, \mathcal{H}_X^1).$

Proof. Let $\alpha \in H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) = H^0(X, a_*^X \mathcal{F}_{\mathbb{Z}})$. Then by (3.3), there exists a Zariski open cover $\{U_i\}$ of X such that $\alpha|_{(U_i)_{an}} = \text{Im}(\beta_i)$ where $\beta_i \in H^1(Y_{an}, \mathbb{Z})$. Therefore $\alpha|_{(U_i)_{an}} \rightarrow 0 \in H^2((U_i)_{an}, \mathbb{Z})$ as shown in the commutative diagram below (where U stands for any of the U_i)

$$\begin{array}{ccccc} H^1(Y_{an}, \mathbb{Z}) & \longrightarrow & H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) & \longrightarrow & H^2(X_{an}, \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1((\pi^{an})^{-1}(U_{an}), \mathbb{Z}) & \longrightarrow & H^0(U_{an}, \mathcal{F}_{\mathbb{Z}}) & \longrightarrow & H^2(U_{an}, \mathbb{Z}) \end{array}$$

This finishes the proof of the lemma. \square

If $\{X_i\}_{i \in I}$ is an admissible family of subsets for the constructible sheaf $R^1\pi_*^{an}\mathbb{Z}$ on X_{an} , then (lemma 2.7) it is also an admissible family for $\mathcal{F}_{\mathbb{Z}}$ and for $a_*^X \mathcal{F}_{\mathbb{Z}} = \mathcal{G}_{\mathbb{Z}}$. We fix such an admissible family once and for all, and fix base points $x_i \in U_i$ with corresponding reduced fibers $F_i = \pi^{-1}(x_i)_{red}$. Let $F = \cup_i F_i = \pi^{-1}(\{x_i \mid i \in I\})$.

By the proper base change theorem, the stalk $(R^1\pi_*^{an}\mathbb{Z})_{x_i}$ is naturally identified with $H^1((F_i)_{an}, \mathbb{Z})$; thus $R^1\pi_*^{an}\mathbb{Z}|_{(U_i)_{an}}$ is a local system with fiber $H^1((F_i)_{an}, \mathbb{Z})$. Note that the stalk $(\mathcal{F}_{\mathbb{Z}})_{x_i}$ has the resulting description

$$(3.4) \quad (\mathcal{F}_{\mathbb{Z}})_{x_i} = \text{image} \left(H^1(Y_{an}, \mathbb{Z}) \rightarrow H^1((F_i)_{an}, \mathbb{Z}) \right).$$

By mixed Hodge theory [D], we deduce that $(\mathcal{F}_{\mathbb{Z}})_{x_i}$ naturally supports a pure Hodge structure of weight 1, which is a quotient Hodge structure of $H^1(Y_{an}, \mathbb{Z})$ (depending only on $i \in I$, and not on the chosen base point $x_i \in U_i$), as well as a Hodge sub-structure of $H^1((F_i)_{an}, \mathbb{Z})$. Finally note also that $\mathcal{F}_{\mathbb{Z}}|_{(U_i)_{an}}$ is a constant sheaf whose fiber supports this pure Hodge structure (i.e., is the underlying lattice).

Lemma 3.2. *$H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ carries a pure Hodge structure of weight one, such that $H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ is a morphism of Hodge structures.*

Proof. From lemma 2.7(iii) there exists an exact sequence of abelian groups

$$0 \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \rightarrow \bigoplus_{i \in I} (\mathcal{F}_{\mathbb{Z}})_{x_i} \rightarrow \bigoplus_{i_0, i_1 \in I, i_0 \leq i_1} (\mathcal{F}_{\mathbb{Z}})_{x_{i_1}}$$

The natural surjective maps $(\mathcal{F}_{\mathbb{Z}})_{x_i} \rightarrow (\mathcal{F}_{\mathbb{Z}})_{x_j}$ (for $i \leq j$) are maps of pure Hodge structures of weight one, which are quotients of $H^1(Y_{an}, \mathbb{Z})$. Hence $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ is identified with the kernel of a morphism of pure Hodge structures of weight 1, and hence itself supports a pure Hodge structure of weight one. Also it is clear from the construction that the composition

$$H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \hookrightarrow \bigoplus_{i \in I} (\mathcal{F}_{\mathbb{Z}})_{x_i}$$

is a direct sum of the natural quotient maps $H^1(Y_{an}, \mathbb{Z}) \rightarrow (\mathcal{F}_{\mathbb{Z}})_{x_i}$, and hence is a morphism of Hodge structures. Hence $H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ is one as well. \square

Proposition 3.3. *$H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \rightarrow H^2(X_{an}, \mathbb{Z})$ is morphism of Hodge structures, i.e., the Hodge structures on $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ and $H^2(X_{an}, \mathbb{Z})$ are compatible.*

Proof. Let $F_i = \pi^{-1}(x_i)$ as above, and let $F = \cup_{i \in I} F_i$. We note that the natural map $H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z}) \rightarrow H^1(F_{an}, \mathbb{Z})$ is an injection (any section in the kernel must vanish in all stalks). This implies that the map $H^2(X_{an}, \mathbb{Z}) \rightarrow H^2(Y_{an}, F_{an}, \mathbb{Z})$ (which is a morphism of mixed Hodge structures) is injective in the following commutative diagram (here $G_i = (\mathcal{F}_{\mathbb{Z}})_{x_i}$).

$$\begin{array}{ccccc}
\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} & \hookrightarrow & \frac{H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} & \hookrightarrow & H^2(X_{an}, \mathbb{Z}) \\
\downarrow & & \nearrow & & \downarrow \\
\frac{\oplus_i G_i}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} & & & & \\
\downarrow & & & & \downarrow \\
\frac{\oplus_i H^1((F_i)_{an}, \mathbb{Z})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} & \hookrightarrow & & \rightarrow & H^2(Y_{an}, F_{an}, \mathbb{Z})
\end{array}$$

We are done, because all the arrows in the above diagram are injections, and the vertical arrows (on the left and right borders), as well as the lower horizontal arrow, are morphisms of mixed Hodge structures. \square

Let $A \subset B$ be an inclusion of abelian groups. Let $A \subset A^s \subset B$ denote the saturation of A in B , i.e., A^s is the smallest subgroup of B containing A such that $\frac{B}{A^s}$ is torsion free. Let $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$ be the saturation of $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})$ in $H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z})$.

Lemma 3.4. $\text{Ker}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) = \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}$.

Proof. It is easy to see, from lemma 3.1, that

$$\text{Ker}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) \supset \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}.$$

We will prove, using lemma 2.8, that given any element

$$\alpha \in \text{ker}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})),$$

and any preimage $\beta_1 \in H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z})$, some non-zero (integral) multiple of β_1 lies in $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \subset H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z})$. This will prove the assertion of the lemma.

Since $\alpha \in H^1(X, \mathcal{H}_X^1)$, there exists a finite Zariski open cover $\{U_i\}$ of X such that $\alpha \mapsto 0$ in $H^2((U_i)_{an}, \mathbb{Z})$ for all i . Let U denote any one of these U_i 's and consider again the above commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
& & & & & & H^1(Y_{an}, \mathbb{Z}) \\
& & & & & & \downarrow \\
& & & & & & H^1(U_{an}, \mathbb{Z}) \longrightarrow H^1(\pi^{-1}(U_{an}), \mathbb{Z}) \\
& & & & & & \downarrow \\
& & & & & & H^2(X_{an}, U_{an}, \mathbb{Z}) \longrightarrow H^2(Y_{an}, \pi^{-1}(U_{an}), \mathbb{Z}) \\
& & & & & & \downarrow \\
& & & & & & H^2(X_{an}, \mathbb{Z}) \longrightarrow H^2(Y_{an}, \mathbb{Z}) \\
& & & & & & \downarrow \\
H^1(Y_{an}, \mathbb{Z}) & \longrightarrow & H^0(X_{an}, R^1\pi_*^{an}\mathbb{Z}) & \longrightarrow & H^2(X_{an}, \mathbb{Z}) & \longrightarrow & H^2(Y_{an}, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(\pi^{-1}(U_{an}), \mathbb{Z}) & \longrightarrow & \Gamma(U_{an}, R^1\pi_*^{an}\mathbb{Z}) & \longrightarrow & H^2(U_{an}, \mathbb{Z}) & \longrightarrow & H^2(\pi^{-1}(U_{an}), \mathbb{Z})
\end{array}$$

We wish to apply lemma 2.8 to this diagram; for this, we need to know that this diagram arises by applying the cohomology functor to a suitable 9-diagram in the category of complexes of abelian groups. It is clear that the above diagram arises by applying the cohomology functor to the following 9-diagram, where all the objects are in the (bounded below) derived category of sheaves of abelian groups on X , and the rows and columns are exact triangles; here K_i are suitable cones.

$$\begin{array}{ccccc}
K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_X & \longrightarrow & R\pi_*\mathbb{Z}_Y & \longrightarrow & C_1 \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
Rj_*\mathbb{Z}_U & \longrightarrow & Rj_*R\pi_*\mathbb{Z}_{\pi^{-1}(U)} & \longrightarrow & Rj_*C_2 \longrightarrow \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

Applying the functor $R\Gamma(X, -)$ yields a 9-diagram in the derived category of abelian groups. Using Cartan-Eilenberg resolutions, this 9-diagram in the derived category is seen to be the image of a 9-diagram where all the objects are complexes of abelian groups and the rows and columns are short exact sequences of complexes. Since the arguments are standard, we omit the details.

Returning to our cohomology diagram, note that the relative cohomology sequences

$$\rightarrow H^1(U_{an}, \mathbb{Z}) \rightarrow H^2(X_{an}, U_{an}, \mathbb{Z}) \rightarrow H^2(X_{an}, \mathbb{Z}) \rightarrow H^2(U_{an}, \mathbb{Z}) \rightarrow$$

and

$$\rightarrow H^1((\pi^{an})^{-1}(U_{an}), \mathbb{Z}) \rightarrow H^2(Y_{an}, (\pi^{an})^{-1}(U_{an}), \mathbb{Z}) \rightarrow H^2(Y_{an}, \mathbb{Z}) \rightarrow H^2((\pi^{an})^{-1}(U_{an}), \mathbb{Z})$$

are sequences in the category of mixed Hodge structures by [D] (8.3.9).

Since $\alpha \rightarrow 0 \in H^2(Y_{an}, \mathbb{Z})$ therefore $\alpha_{\mathbb{Q}} \in W_1 H^2(X_{an}, \mathbb{Q})$ by [D], Proposition 8.2.5. This implies, by [D], Theorem 2.3.5 (*i.e.*, strictness of morphisms of mixed Hodge structures with respect to W) that we can choose $\beta_2 \in H^2(X_{an}, U_{an}, \mathbb{Z})$ such that

$$(\beta_2)_{\mathbb{Q}} \in W_1 H^2(X_{an}, U_{an}, \mathbb{Q}), \quad \beta_2 \mapsto n\alpha, \quad n \in \mathbb{Z}_{>0}.$$

Let

$$\beta_2 \mapsto \gamma_2 \in H^2(Y_{an}, (\pi^{an})^{-1}(U_{an}), \mathbb{Z}) \cong \mathbb{Z}(-1)^k,$$

for some $k \geq 0$, where the last isomorphism is because Y is non-singular; then

$$(\gamma_2)_{\mathbb{Q}} \in W_1 H^2(Y_{an}, (\pi^{an})^{-1}(U_{an}), \mathbb{Q}) = 0,$$

i.e., $\gamma_2 = 0$. So we can choose a preimage $\delta_2 \in H^1((\pi^{an})^{-1}(U_{an}), \mathbb{Z})$ of γ_2 to be zero. On the other hand, chasing the diagram the other way, we get $n\beta_1 \in H^0(X_{an}, R^1 \pi_*^{an} \mathbb{Z})$ which lifts $n\alpha$, and $n\beta \mapsto n\gamma_1 \in H^0(U_{an}, R^1 \pi_*^{an} \mathbb{Z})$; now take a lift $n\delta_1 \in H^1((\pi^{an})^{-1}(U_{an}), \mathbb{Z})$ of $n\gamma_1$.

By lemma 2.8, we know that $n\delta_1 \equiv \delta_2 = 0$ modulo the images of $H^1(Y_{an}, \mathbb{Z})$ and $H^1(U_{an}, \mathbb{Z})$. Therefore

$$n\gamma_1 \in \text{Im}(H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(U_{an}, R^1 \pi_*^{an} \mathbb{Z})).$$

This proves that $n\beta_1|_{U_{an}}$ comes from $H^1(Y_{an}, \mathbb{Z})$. Since X has a finite cover by such open sets U , we see that $\beta_1 \in H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$. \square

Corollary 3.5. *There exists a short exact sequence of mixed Hodge structures*

$$0 \rightarrow \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \rightarrow H^1(X, \mathcal{H}_X^1) \rightarrow \text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) \rightarrow 0.$$

Let $\text{Im}(H^1(Y_{an}, \mathbb{Z}))^s$ denote the saturation of $H^1(Y_{an}, \mathbb{Z})$ in $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$. Then

$$\frac{\text{Im}(H^1(Y_{an}, \mathbb{Z}))^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} = \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \right)_{\text{torsion}}.$$

Since $NS(X) \subset F^1 H^2(X_{an}, \mathbb{Z})$ it follows that it has finite intersection with $\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}$ which is a pure Hodge structure of weight one. On the other hand, $H^2(X_{an}, \mathbb{Z})_{\text{torsion}} \subset NS(X)$ from the exponential sequence. Thus,

$$\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \right)_{\text{torsion}} = NS(X) \cap \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}.$$

Hence we get the exact sequence

$$0 \rightarrow \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))^s} \rightarrow \frac{H^1(X, \mathcal{H}_X^1)}{NS(X)} \xrightarrow{f} \frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{\text{Im}(NS(X))} \rightarrow 0, \quad (+)$$

It is clear that the third term is pure of type $(1, 1)$ as it lies inside $\frac{NS(Y)}{\text{Im}(NS(X))}$.

Let $A = \frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{\text{Im}(NS(X))}$ and let

$$M = f^{-1}(A_{\text{torsion}}).$$

We then have a short exact sequence of mixed Hodge structures

$$0 \rightarrow M \rightarrow \frac{H^1(X, \mathcal{H}_X^1)}{NS(X)} \rightarrow \frac{A}{A_{\text{torsion}}} \cong \mathbb{Z}(-1)^r \rightarrow 0, \quad (++)$$

where the third term is free of rank r and pure of type $(1, 1)$, and M is a pure Hodge structure of weight 1. Further, all of the underlying abelian groups are free.

We recall some facts about extensions of mixed Hodge structures (see [C], for example). Let H be a finitely generated abelian group which supports a pure Hodge structure of weight one, and G a finitely generated abelian group, regarded as a pure Hodge structure of type $(0, 0)$. Then there is a natural identification of the abelian group $\text{Ext}_{\mathbf{MHS}}^1(G(-1), H)$ of extensions in the category \mathbf{MHS} of mixed Hodge structures with the abelian group $\text{Hom}(G, J(H))$, where

$$J(H) = J^1(H) = \frac{H_{\mathbb{C}}}{F^1 H_{\mathbb{C}} + \text{Im}(H)};$$

here $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$ and F gives the Hodge filtration. In particular we have

$$\text{Ext}_{\mathbf{MHS}}^1(\mathbb{Z}(-1), H) = J(H).$$

If

$$0 \rightarrow H \rightarrow E \rightarrow G(-1) \rightarrow 0$$

is an extension of mixed Hodge structures, let $\psi_E : G \rightarrow J(H)$ be the corresponding homomorphism (which we call the *extension class map* of E). This may be described as follows: there is an identification

$$\alpha : \frac{H_{\mathbb{C}}}{F^1 H_{\mathbb{C}}} \xrightarrow{\cong} \frac{E_{\mathbb{C}}}{F^1 E_{\mathbb{C}}},$$

giving

$$\beta : J(H) = \frac{H_{\mathbb{C}}}{F^1 H_{\mathbb{C}} + H} \xrightarrow{\cong} \frac{E_{\mathbb{C}}}{F^1 E_{\mathbb{C}} + H},$$

and ψ_E is the composition

$$G \cong \frac{E}{H} \rightarrow \frac{E_{\mathbb{C}}}{F^1 E_{\mathbb{C}} + H} \xrightarrow{\beta^{-1}} J(H).$$

In case $G = \mathbb{Z}^{\oplus r}$ is free abelian, we have that $\text{Hom}_{\mathbf{MHS}}(\mathbb{Z}(-1), G)$ is naturally identified with $\ker \psi_E$. Also, if G is free abelian and H is polarizable, then $J(H)$ is an abelian variety, and for an extension E , the homomorphism $\psi_E : G \rightarrow J(H)$ is the 1-motive over \mathbb{C} associated to the mixed Hodge structure E by the procedure in [D], (10.1.3).

In particular, the sequence of mixed Hodge structures $(+)$ is an extension of a pure Hodge structure of type $(1, 1)$ by a pure weight one Hodge structure and hence gives rise to an extension class homomorphism,

$$\psi : \frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{NS(X)} \rightarrow J \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{(\text{Im}(H^1(Y_{an}, \mathbb{Z})))^s} \right).$$

Similarly, the sequence of mixed Hodge structures $(++)$ gives rise to a related homomorphism

$$\psi_1 : \mathbb{Z}^{\oplus r} \cong \frac{A}{A_{tors}} \rightarrow J(M),$$

which is in fact a 1-motive. Note that $J(M)$ is isogenous to $J \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{(\text{Im}(H^1(Y_{an}, \mathbb{Z})))^s} \right)$

which in turn is isogenous to $J \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \right)$.

By the above remarks, our main result Theorem 1.1 is equivalent to proving ψ_1 is injective.

4. PROOF OF THE MAIN THEOREM

4.1. Construction of a 1-motive. The aim of this section is to directly construct a certain 1-motive over \mathbb{C} . The proof of the Main Theorem will be by showing that it is isogenous to that associated to $(H^1(X, \mathcal{H}_X^1)/NS(X)) \otimes \mathbb{Z}(1)$.

Let $\pi : Y \rightarrow X$ be a desingularization of X as before and let $U \subset X$ be a Zariski open subset. We have an exact sequence of groups

$$\text{Pic}^0(Y) \rightarrow \text{Pic}(\pi^{-1}(U)) \rightarrow H^1(\pi^{-1}(U), \mathcal{H}_Y^1) \rightarrow 0.$$

We sheafify this on $X = X_{zar}$ to get an exact sequence of sheaves

$$\text{Pic}^0(Y)_X \rightarrow R^1\pi_*\mathcal{O}_Y^* \rightarrow R^1\pi_*\mathcal{H}_Y^1 \rightarrow 0$$

where $\text{Pic}^0(Y)_X$ is the constant sheaf on X associated to the group $\text{Pic}^0(Y)$. Define \mathcal{F} to be the Zariski sheaf

$$\mathcal{F} := \text{Im}(\text{Pic}^0(Y)_X \rightarrow R^1\pi_*\mathcal{O}_Y^*)$$

on X . Hence we have short exact sequence of sheaves

$$(4.1) \quad 0 \rightarrow \mathcal{F} \rightarrow R^1\pi_*\mathcal{O}_Y^* \rightarrow R^1\pi_*\mathcal{H}_Y^1 \rightarrow 0$$

Lemma 4.1. (1) *There is an injective map*

$$\mu : J(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s) \rightarrow H^0(X, \mathcal{F}),$$

whose image $H^0(X, \mathcal{F})^0$ is a subgroup of finite index, such that the natural map $\text{Pic}^0(Y) \rightarrow H^0(X, \mathcal{F})$ factors through μ . The induced map $\text{Pic}^0(Y) \rightarrow H^0(X, \mathcal{F})^0$ is that determined by the map on Hodge structures $H^1(Y_{an}, \mathbb{Z}) \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$. Thus $H^0(X, \mathcal{F})^0$ is the group of \mathbb{C} -points of an abelian variety, such that $\text{Pic}^0(Y) \rightarrow H^0(X, \mathcal{F})^0$ is a homomorphism of abelian varieties.

(2) $J \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))} \right)$ is isogenous to $\frac{H^0(X, \mathcal{F})^0}{\text{Im}(\text{Pic}^0(Y))}$, and hence the latter is isogenous to $J(M)$.

Proof. Let $x \in X$ be any point and $F_x = \pi^{-1}(x)_{red}$. Let $Y_x = \text{Spec}(\mathcal{O}_{X,x}) \times_X Y$ and $F_x^n = \text{Spec}\left(\frac{\mathcal{O}_{X,x}}{\mathcal{M}_x^n}\right) \times_X Y$, where $\mathcal{M} \subset \mathcal{O}_{X,x}$ is the maximal ideal. For each n we have the restriction maps $h_n : \text{Pic}(F_x^n) \rightarrow \text{Pic}(F_x)$.

We claim that the kernel of h_n , for each n , is a \mathbb{C} -vector space. To see this consider the short exact sequence of Zariski sheaves

$$0 \longrightarrow \mathcal{I}_{F_x^n/F_x} \xrightarrow{\exp} \mathcal{O}_{F_x^n}^* \xrightarrow{h_n} \mathcal{O}_{F_x}^* \longrightarrow 0$$

where \exp denotes the exponential map, which makes sense as $\mathcal{I}_{F_x^n/F_x}$ is nilpotent. Considering the associated cohomology sequence we get

$$0 \rightarrow H^1(F_x, \mathcal{I}_{F_x^n/F_x}) \rightarrow \text{Pic}(F_x^n) \xrightarrow{h_n} \text{Pic}(F_x)$$

which proves the kernel of h_n , for each n , is a \mathbb{C} -vector space.

For each n and for each $x \in X$, we have a commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{f_n} & & \\ & \text{Pic}^0(Y) & \longrightarrow & \text{Pic}(Y_x) & \longrightarrow & \text{Pic}(F_x^n) \\ & \searrow g & & \downarrow & \swarrow h_n \\ & & & \text{Pic}(F_x) & \end{array}$$

Let f_n be the composition $\text{Pic}^0(Y) \rightarrow \text{Pic}(Y_x) \rightarrow \text{Pic}(F_x^n)$. Then, $\text{Ker}(f_n)$ and $\text{Ker}(g)$ are both closed subgroups of $\text{Pic}^0(Y)$ hence are compact (topological) groups. Since $\text{Ker}(h_n)$ is a \mathbb{C} -vector space it follows that $\text{Ker}(f_n) = \text{Ker}(g)$, as any continuous homomorphism from a compact group to a \mathbb{C} -vector space is zero (note that $\text{Pic}^0(Y)$, $\text{Pic}^0(F_x^n)$, and $\text{Pic}^0(F_x)$ are isomorphic to the corresponding analytic groups, by GAGA, and hence from the exponential sequence carry natural topologies, such that the restriction homomorphisms are continuous).

Passing to the inverse limit we have a commutative diagram

$$\begin{array}{ccccc} \text{Pic}^0(Y) & \xrightarrow{f} & \text{Pic}(Y_x) & \hookrightarrow & \text{Pic}(\hat{Y}_x) \\ & \searrow g & \downarrow & \swarrow h & \\ & & \text{Pic}(F_x) & & \end{array}$$

where \hat{Y}_x stands for the completion of Y_x along F_x . By Grothendieck's Formal Function Theorem [H, Ch.III, Th.11.1] and the fact that $\text{Pic}(\hat{Y}_x) \rightarrow \varprojlim_n (\text{Pic}(F_x^n))$

is an isomorphism [H, Ch.II, Ex.9.6], we have that $\text{Pic}(Y_x) \rightarrow \text{Pic}(\hat{Y}_x)$ is an injection. Thus it follows that $\text{Ker}(f) = \text{Ker}(g)$. We have from the definition of \mathcal{F} that the stalk of \mathcal{F} at x , $\mathcal{F}_x = \text{Im}(\text{Pic}^0(Y) \rightarrow \text{Pic}(Y_x))$. By our analysis so far we have proved that the natural map $\mathcal{F}_x \rightarrow \text{Pic}(F_x)$ is an inclusion, and it clearly factors through the the subgroup $\text{Pic}^0(F_x)$.

By the results of Du Bois [DB] there exists a commutative triangle

$$\begin{array}{ccc} H^1(F_x, \mathbb{C}) & \longrightarrow & H^1(F_x, \mathcal{O}_{F_x}) \\ & \searrow & \downarrow \alpha \\ & & \frac{H^1(F_x, \mathbb{C})}{F^1 H^1(F_x, \mathbb{C})} \end{array}$$

Note that $Ker(\alpha)$ is a \mathbb{C} -vector space. Now α induces a map

$$\beta : Pic^0(F_x) \cong \frac{H^1(F_x, \mathcal{O})}{H^1(F_x, \mathbb{Z})} \rightarrow J(H^1(F_x, \mathbb{Z})).$$

Thus we have a diagram

$$\begin{array}{ccc} H^1(F_x, \mathcal{O}_{F_x}) & \xrightarrow{\alpha} & \frac{H^1(F_x, \mathbb{C})}{F^1 H^1(F_x, \mathbb{C})} \\ \downarrow & & \downarrow \\ Pic^0(F_x) & \xrightarrow{\beta} & J(H^1(F_x, \mathbb{Z})) \end{array}$$

Since $H^1(F_x, \mathbb{Z})$ is a mixed Hodge structure with weights 0 and 1 (by [D2], as F_x is a projective variety), $H^1(F_x, \mathbb{Z})$ injects into $\frac{H^1(F_x, \mathbb{C})}{F^1 H^1(F_x, \mathbb{C})}$. Thus it is clear from the above diagram $Ker(\beta) = Ker(\alpha)$ and so $Ker(\beta)$ is also a \mathbb{C} -vector space. Hence the composite $\mathcal{F}_x \rightarrow Pic^0(F_x) \rightarrow J(H^1(F_x, \mathbb{Z}))$ is injective, as \mathcal{F}_x is a compact group, from its definition.

Let $\mathcal{F}_{\mathbb{Z},x} = Im(H^1(Y, \mathbb{Z}) \rightarrow H^1(F_x, \mathbb{Z}))$ be the stalk of $\mathcal{F}_{\mathbb{Z}}$ at x . Let $\mathcal{F}_{\mathbb{Z},x}^s$ be the saturation of $\mathcal{F}_{\mathbb{Z},x}$ in $H^1(F_x, \mathbb{Z})$. The inclusion $\mathcal{F}_{\mathbb{Z},x} \rightarrow H^1(F_x, \mathbb{Z})$ induces a natural map with finite kernel $J(\mathcal{F}_{\mathbb{Z},x}) \rightarrow J(H^1(F_x, \mathbb{Z}))$. In fact this map factors as

$$J(\mathcal{F}_{\mathbb{Z},x}) \longrightarrow J(\mathcal{F}_{\mathbb{Z},x}^s) \longrightarrow J(H^1(F_x, \mathbb{Z})).$$

The second map is an inclusion and $J(\mathcal{F}_{\mathbb{Z},x}^s)$ is the image of $J(\mathcal{F}_{\mathbb{Z},x})$ in $J(H^1(F_x, \mathbb{Z}))$.

We thus have a commutative diagram with surjective and injective maps as follows (where we identify $Pic^0(Y)$ with $J(H^1(Y_{an}, \mathbb{Z}))$).

$$\begin{array}{ccccc} Pic^0(Y) & \longrightarrow & J(\mathcal{F}_{\mathbb{Z},x}) & \longrightarrow & J(\mathcal{F}_{\mathbb{Z},x}^s) \\ \downarrow & & \downarrow & \nearrow & \\ \mathcal{F}_x & \hookrightarrow & J(H^1(F_x, \mathbb{Z})) & & \end{array}$$

Since it is clear from the diagram that $J(\mathcal{F}_{\mathbb{Z},x}^s)$ and \mathcal{F}_x are both the image of $Pic^0(Y)$ in $J(H^1(F_x, \mathbb{Z}))$ it follows that $J(\mathcal{F}_{\mathbb{Z},x}^s) \cong \mathcal{F}_x$. Therefore there exists a map $J(\mathcal{F}_{\mathbb{Z},x}) \rightarrow \mathcal{F}_x$ which is an isogeny.

We had proved that the sheaf $\mathcal{F}_{\mathbb{Z}}$ was constructible, i.e., constant with groups G_i over locally closed sets $(U_i)_{an}$, and this data gives rise to a flasque resolution of $a_* \mathcal{F}_{\mathbb{Z}}$ in the Zariski site (by lemma 2.7 and lemma 2.5). It is then clear that analogous results hold also for the sheaf $\mathcal{F}_{\mathbb{Z}}^s$ where $\mathcal{F}_{\mathbb{Z}}^s$ denotes the saturation of

the sheaf $\mathcal{F}_{\mathbb{Z}}$ in $R^1\pi_*^{an}\mathbb{Z}$ (which is a torsion-free sheaf). Thus the abelian varieties $J(\mathcal{F}_{\mathbb{Z},x}^s)$ are constant quotients of $Pic^0(Y)$ over the strata U_i , hence so are \mathcal{F}_x . This proves that the sheaf \mathcal{F} is a constructible sheaf on X for the Zariski topology, with admissible family $\{X_i\}$, and further (by lemma 2.5) \mathcal{F} has a flasque resolution similar to $a_*\mathcal{F}_{\mathbb{Z}}$.

Taking global sections of the flasque resolution of $a_*\mathcal{F}_{\mathbb{Z}}^s$, we get an exact sequence

$$0 \rightarrow H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}^s) \rightarrow \bigoplus_i \mathcal{F}_{\mathbb{Z},x_i}^s \rightarrow \bigoplus_{i<j} \mathcal{F}_{\mathbb{Z},x_j}^s.$$

Also it is clear from the definitions that $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}^s) = H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$. Applying J on all the terms, we obtain a complex

$$0 \rightarrow J(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}^s)) \rightarrow \bigoplus_i J(\mathcal{F}_{\mathbb{Z},x_i}^s) \rightarrow \bigoplus_{i<j} J(\mathcal{F}_{\mathbb{Z},x_j}^s)$$

This complex is exact on the left and has finite homology in the middle, since $H^0(\bigoplus_{i<j} \mathcal{F}_{\mathbb{Z},x_j}^s)$ is torsion-free. Similarly taking global sections of the flasque resolution of \mathcal{F} we get an exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow \bigoplus_i \mathcal{F}_{x_i} \rightarrow \bigoplus_{i<j} \mathcal{F}_{x_j}.$$

There exists a commutative diagram

$$\begin{array}{ccccc} J(H^0(X, \mathcal{F}_{\mathbb{Z}})^s) & \longrightarrow & \bigoplus_i J(\mathcal{F}_{\mathbb{Z},x_i}^s) & \longrightarrow & \bigoplus_{i<j} J(\mathcal{F}_{\mathbb{Z},x_j}^s) \\ \mu \downarrow \vdots & & \parallel & & \parallel \\ 0 \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & \bigoplus_i \mathcal{F}_{x_i} & \longrightarrow & \bigoplus_{i<j} \mathcal{F}_{x_j} \end{array}$$

where the two vertical arrows are isomorphisms. Hence the dotted arrow μ exists, and is an inclusion with finite cokernel.

Define

$$H^0(X, \mathcal{F})^0 = \text{Im}(J(H^0(X, \mathcal{F}_{\mathbb{Z}})^s)).$$

Clearly this is an abelian variety, and there is an isogeny $J(H^0(X, \mathcal{F}_{\mathbb{Z}})) \rightarrow H^0(X, \mathcal{F})^0$. Also, by construction, the natural map $\text{Pic}^0(Y) \rightarrow H^0(X, \mathcal{F})$ clearly factors through the map

$$\text{Pic}^0(Y) = J(H^1(Y_{an}, \mathbb{Z})) \rightarrow J(H^0(X, \mathcal{F}_{\mathbb{Z}})).$$

Thus we have an isogeny $\frac{J(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}))}{\text{Im}(\text{Pic}^0(Y))} \rightarrow \frac{H^0(X, \mathcal{F})^0}{\text{Im}(\text{Pic}^0(Y))}$. We finally note that there exists an isogeny

$$\frac{J(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}))}{\text{Im}(\text{Pic}^0(Y))} \rightarrow J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}\right)$$

since $J(H^1(Y_{an}, \mathbb{Z})) \cong \text{Pic}^0(Y)$.

This finishes the proof that $\frac{H^0(X, \mathcal{F})^0}{\text{Im}(\text{Pic}^0(Y))}$ and $J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})}{\text{Im}(H^1(Y_{an}, \mathbb{Z}))}\right)$ are isogenous. \square

We can now construct a 1-motive, as follows. Since X is normal, we have that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, and so we have an exact sequence

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(Y) \rightarrow H^0(X, R^1\pi_*\mathcal{O}_Y^*).$$

This induces another exact sequence

$$NS(X) \rightarrow NS(Y) \rightarrow \frac{H^0(X, R^1\pi_*\mathcal{O}_Y^*)}{\text{Im}(\text{Pic}^0(Y))}.$$

We thus have an *injective* map

$$(4.2) \quad \frac{NS(Y)}{\text{Im}(NS(X))} \rightarrow \frac{H^0(X, R^1\pi_*\mathcal{O}_Y^*)}{\text{Im}(\text{Pic}^0(Y))}$$

Lemma 4.2. *The map (4.2) induces an (injective) map*

$$\phi : \frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{\text{Im}(NS(X))} \rightarrow \frac{\Gamma(X, \mathcal{F})}{\text{Im}(\text{Pic}^0(Y))}.$$

Proof. Using the short exact sequence of sheaves (4.1), we get the following commutative diagram, whose right column is exact,

$$\begin{array}{ccc} \frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{\text{Im}(NS(X))} & \xrightarrow{\dots\dots\dots \phi} & \frac{\Gamma(X, \mathcal{F})}{\text{Im}(\text{Pic}^0(Y))} \\ \downarrow & & \downarrow \\ \frac{NS(Y)}{\text{Im}(NS(X))} & \xrightarrow{\hookrightarrow} & \frac{\Gamma(X, R^1\pi_*\mathcal{O}^*)}{\text{Im}(\text{Pic}^0(Y))} \\ & & \downarrow \\ & & \Gamma(X, R^1\pi_*\mathcal{H}_Y^1) \end{array}$$

Here, we claim the dotted arrow ϕ exists (and is also injective) because the composition

$$\frac{\text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{NS(X)} \rightarrow H^0(X, R^1\pi_*\mathcal{H}_Y^1)$$

is zero. This is obvious as this map can be described in the following way: given the image in $H^1(Y, \mathcal{H}_Y^1) = NS(Y)$ of a Zariski locally trivial cohomology class $\eta \in H^1(X, \mathcal{H}_X^1)$, consider a line bundle L_η on Y which represents it, then consider the line bundle restricted to open sets $\pi^{-1}(U) \subset Y$, $L_\eta|_{\pi^{-1}(U)}$, (where $U \subset X$ open) and take the Chern classes of these restrictions. These give a global section of $R^1\pi_*\mathcal{H}_Y^1$ which is zero as the line bundle came from a locally trivial cohomology class on X . \square

Since $\frac{\Gamma(X, \mathcal{F})}{\text{Im}(\text{Pic}^0(Y))}$ has a subgroup of finite index which is an abelian variety, ϕ determines a 1-motive in an obvious way,

$$B \rightarrow \frac{\Gamma(X, \mathcal{F})^0}{\text{Im}(\text{Pic}^0(Y))},$$

where B is the inverse image under ϕ of the abelian variety.

Remark 4.3. We do not know if $\Gamma(X, \mathcal{F})$ is itself an abelian variety, *i.e.*, if $\Gamma(X, \mathcal{F})^0 = \Gamma(X, \mathcal{F})$.

4.2. Comparison of the two 1-motives. We now finish the proof of the theorem, by comparing the 1-motive constructed above using ϕ with that constructed earlier, using the extension class map ψ for the mixed Hodge structure on $H^1(X, \mathcal{H}_X^1)$.

Recall that $\{X_i\}_{i \in I}$ is the chosen admissible family of subsets for $R^1\pi_*^{an}\mathbb{Z}$, and hence for $\mathcal{F}_{\mathbb{Z}}$ and \mathcal{F} as well; recall also the corresponding (irreducible, non-singular) locally closed strata $\{U_i\}_{i \in I}$. Also recall the choice of points $x_i \in U_i$, and $F_i = \pi^{-1}(x_i)$, $F = \cup_{i \in I} F_i$.

Suitably blow up Y to get $f: \tilde{Y} \rightarrow Y$, with \tilde{Y} non-singular projective, so that the reduced strict transform of F is a *smooth* possibly disconnected subvariety \tilde{F} . Note that there exists the following diagram.

$$\begin{array}{ccc} \tilde{F} & \hookrightarrow & \tilde{Y} \\ \downarrow & & \downarrow f \\ F & \hookrightarrow & Y \end{array}$$

Now consider the following commutative diagram which is a diagram in the category of mixed Hodge structures

$$\begin{array}{ccccccc} 0 \longrightarrow & \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))} & \longrightarrow & H^1(X, \mathcal{H}_X^1) & \longrightarrow & Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \frac{H^1(F_{an}, \mathbb{Z})}{Im(H^1(Y_{an}, \mathbb{Z}))} & \longrightarrow & H^2(Y_{an}, F_{an}, \mathbb{Z}) & \longrightarrow & Im(H^2(Y_{an}, F_{an}, \mathbb{Z}) \rightarrow H^2(Y_{an}, \mathbb{Z})) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \frac{H^1(\tilde{F}_{an}, \mathbb{Z})}{Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))} & \longrightarrow & H^2(\tilde{Y}_{an}, \tilde{F}_{an}, \mathbb{Z}) & \longrightarrow & Im(H^2(\tilde{Y}_{an}, \tilde{F}_{an}, \mathbb{Z}) \rightarrow H^2(\tilde{Y}_{an}, \mathbb{Z})) & \longrightarrow 0 \end{array}$$

Let $Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))^s$ be the saturation of $Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))$ in $H^1(\tilde{F}_{an}, \mathbb{Z})$. Then, arguing as before,

$$Im(NS(X)) \cap \frac{H^1(\tilde{F}_{an}, \mathbb{Z})}{Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))} = \frac{Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))^s}{Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))}.$$

So we have a short exact sequence of mixed Hodge structures

$$0 \longrightarrow \frac{H^1(\tilde{F}_{an}, \mathbb{Z})}{Im(H^1(\tilde{Y}_{an}, \mathbb{Z}))^s} \longrightarrow \frac{H^2(\tilde{Y}_{an}, \tilde{F}_{an}, \mathbb{Z})}{Im(NS(X))} \longrightarrow \frac{Im(H^2(\tilde{Y}_{an}, \tilde{F}_{an}, \mathbb{Z}) \rightarrow H^2(\tilde{Y}_{an}, \mathbb{Z}))}{Im(NS(X))} \longrightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s} & \longrightarrow & \frac{H^1(X, \mathcal{H}_X^1)}{NS(X)} & \longrightarrow & \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s} & \longrightarrow & \frac{H^2(\widetilde{Y}_{an}, \widetilde{F}_{an}, \mathbb{Z})}{Im(NS(X))} & \longrightarrow & \frac{Im(H^2(\widetilde{Y}_{an}, \widetilde{F}_{an}, \mathbb{Z}) \rightarrow H^2(\widetilde{Y}_{an}, \mathbb{Z}))}{Im(NS(X))} \longrightarrow 0
\end{array}$$

The following diagram commutes by functoriality of the extension class maps.

$$\begin{array}{ccc}
Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) & \xrightarrow{\psi} & J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) \\
\downarrow & & \downarrow \\
Im(H^2(\widetilde{Y}_{an}, \widetilde{F}_{an}, \mathbb{Z}) \rightarrow H^2(\widetilde{Y}_{an}, \mathbb{Z})) & \longrightarrow & J\left(\frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s}\right)
\end{array}$$

where

$$J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) \rightarrow J\left(\frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s}\right)$$

is induced from the map on the underlying Hodge structures.

Let

$$\psi' : Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z})) \rightarrow J\left(\frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s}\right)$$

be the composite in the diagram above.

We also have a natural “sheaf theoretic” map

$$\phi' : \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{NS(X)} \rightarrow J\left(\frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s}\right)$$

which is defined as follows. Let $\eta \in Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))$. Consider a line bundle L on Y such that $c_1(L) = \eta$. Then, $L|_F$ (where $F = \cup_i F_i$) gives an element of $Pic^0(F)$ and hence an element of $J(H^1(F, \mathbb{Z}))$ via the mapping $Pic^0(F) \rightarrow J(H^1(F, \mathbb{Z}))$. Under this mapping $NS(X)$ goes to zero, hence we get a well defined mapping

$$\frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{NS(X)} \rightarrow J\left(\frac{H^1(F_{an}, \mathbb{Z})}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right).$$

Now compose with the map (induced by the morphism of the underlying Hodge structures)

$$J\left(\frac{H^1(F_{an}, \mathbb{Z})}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) \rightarrow J\left(\frac{H^1(\widetilde{F}_{an}, \mathbb{Z})}{Im(H^1(\widetilde{Y}_{an}, \mathbb{Z}))^s}\right)$$

to get

$$\phi' : \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{NS(X)} \rightarrow J \left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))} \right).$$

It is easy to see the following diagram commutes

$$\begin{array}{ccc} \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))} & \xrightarrow{\phi} & \frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} \\ & \searrow \phi' & \downarrow \\ & & J \left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))} \right) \end{array}$$

where the map $\frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} \rightarrow J \left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))} \right)$ is the composition

$$\frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} \rightarrow \frac{Pic^0(F)}{Im(Pic^0(Y))} \rightarrow \frac{Pic^0(\widetilde{F})}{Im(Pic^0(\widetilde{Y}))} \rightarrow J \left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))} \right).$$

We now note that the map $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s \rightarrow H^1(F_{an}, \mathbb{Z})$ is an injective map. Since $W_0 H^1(F_{an}, \mathbb{Q}) = Ker(H^1(F_{an}, \mathbb{Q}) \rightarrow H^1(\widetilde{F_{an}}, \mathbb{Q}))$ and $H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s$ is pure of weight one, it follows that

$$(Ker(H^1(F_{an}, \mathbb{Q}) \rightarrow H^1(\widetilde{F_{an}}, \mathbb{Q}))) \cap Im(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \rightarrow H^1(F_{an}, \mathbb{Q})) = 0.$$

Hence the composite

$$H^0(X_{an}, \mathcal{F}_{\mathbb{Z}}) \rightarrow H^1(F_{an}, \mathbb{Z}) \rightarrow H^1(\widetilde{F_{an}}, \mathbb{Z})$$

has finite kernel, and is hence injective (as $\mathcal{F}_{\mathbb{Z}} \subset R^1 \pi_*^{an} \mathbb{Z}$ is torsion-free). It follows that

$$J \left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s} \right) \rightarrow J \left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s} \right), \quad (+ + +)$$

has finite kernel.

We have the following diagram which shows all the maps we have constructed so far (the outer border is not yet known to commute).

$$\begin{array}{ccccc}
J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) & \xleftarrow{\psi} & \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))} & & \\
\downarrow & & \downarrow \phi & & \\
& & \frac{J(H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s)}{Im(Pic^0(Y))} \hookrightarrow \frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} & & \\
& & \downarrow & & \\
J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right) & \xleftarrow{\quad} & \frac{J(H^1(\widetilde{F_{an}}, \mathbb{Z}))}{Im(Pic^0(\widetilde{Y}))} \xleftarrow{\quad} & \frac{Pic^0(\widetilde{F})}{Im(Pic^0(\widetilde{Y}))} &
\end{array}$$

We will prove that the following subdiagram commutes

$$\begin{array}{ccc}
J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) & \xleftarrow{\psi} & \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))} \\
\downarrow & & \downarrow \phi \\
J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right) & \xleftarrow{\quad} & \frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))}
\end{array} \quad (*)$$

Note that the composite map

$$\frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(Y))} \rightarrow \frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} \rightarrow J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right)$$

is the previously defined map ϕ' , and the map

$$\frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(Y))} \rightarrow J\left(\frac{H^0(X_{an}, \mathcal{F}_{\mathbb{Z}})^s}{Im(H^1(Y_{an}, \mathbb{Z}))^s}\right) \rightarrow J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right)$$

is our previously defined map ψ' . Thus the commutativity of the above diagram is equivalent to proving

$$\psi' = \phi'.$$

Assuming this diagram commutes we finish the proof of Theorem 1.1 as follows. We claim that the composite map

$$\frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))} \rightarrow J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right)$$

has finite kernel, as ϕ is injective and the map

$$\frac{H^0(X, \mathcal{F})}{Im(Pic^0(Y))} \rightarrow J\left(\frac{H^1(\widetilde{F_{an}}, \mathbb{Z})}{Im(H^1(\widetilde{Y_{an}}, \mathbb{Z}))^s}\right)$$

has finite kernel (combining lemma 4.1 with the fact that the map in $(+++)$ above has finite kernel). Thus, ψ has finite kernel. Now recall the map

$$\psi_1 : \mathbb{Z}^r \cong \frac{A}{A_{tors}} \rightarrow J(M),$$

where $A = \frac{Im(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))}{Im(NS(X))}$. It follows immediately that ψ_1 has finite kernel. Since $\frac{A}{A_{tors}}$ is a free abelian group, it follows that ψ_1 is injective. This is equivalent to proving our main result, Theorem 1.1, as has been remarked before.

We now finish the final part of the proof by showing the commutativity of the diagram $(*)$. Let Z be a smooth projective variety over \mathbb{C} and $W \subset Z$ be a smooth subvariety. Let $\eta \in H^2(Z_{an}, \mathbb{Z})$ be an algebraic class (i.e., let $\eta \in NS(Z)$), such that $\eta \mapsto 0 \in H^2(W_{an}, \mathbb{Z})$. Then, η gives rise to the following pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} & \longrightarrow & H^2(Z_{an}, W_{an}, \mathbb{Z}) & \longrightarrow & Ker(H^2(Z_{an}, \mathbb{Z}) \rightarrow H^2(W_{an}, \mathbb{Z})) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} & \longrightarrow & B & \longrightarrow & \mathbb{Z}[\eta] \longrightarrow 0 \end{array}$$

Thus we have an extension class map

$$\mathbb{Z} \cong \mathbb{Z}[\eta] \xrightarrow{\psi''} J\left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})}\right).$$

Again given η as above consider L_η , a line bundle on Z which has Chern class η . Restrict this line bundle on W to get $L_\eta|_W \in Pic^0(W_{an}) \cong J(H^1(W_{an}, \mathbb{Z}))$. This gives us a well-defined mapping

$$\mathbb{Z} \cong \mathbb{Z}[\eta] \xrightarrow{\phi''} J\left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})}\right).$$

We now have the following lemma.

Lemma 4.4. *With the above notation $\psi'' = \phi''$, i.e., the extension class map and the restriction map corresponding to the class η are the same.*

Proof. Consider the following diagram with exact rows and columns

$$\begin{array}{ccccccc}
\longrightarrow & H^0(W_{an}, \mathbb{Z}) & \longrightarrow & H^1(Z_{an}, W_{an}, \mathbb{Z}) & \longrightarrow & H^1(Z_{an}, \mathbb{Z}) & \longrightarrow & H^1(W_{an}, \mathbb{Z}) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \frac{H^0(W_{an}, \mathbb{C})}{F^1 H^0(W_{an}, \mathbb{C})} & \longrightarrow & \frac{H^1(Z_{an}, W_{an}, \mathbb{C})}{F^1 H^1(Z_{an}, W_{an}, \mathbb{C})} & \longrightarrow & \frac{H^1(Z_{an}, \mathbb{C})}{F^1 H^1(Z_{an}, \mathbb{C})} & \longrightarrow & \frac{H^1(W_{an}, \mathbb{C})}{F^1 H^1(W_{an}, \mathbb{C})} & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^0(W_{an}, \mathcal{O}_{W_{an}}^*) & \longrightarrow & \text{Pic}(Z_{an}, W_{an}) & \longrightarrow & \text{Pic}(Z_{an}) & \longrightarrow & \text{Pic}(W_{an}) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^1(W_{an}, \mathbb{Z}) & \longrightarrow & H^2(Z_{an}, W_{an}, \mathbb{Z}) & \longrightarrow & H^2(Z_{an}, \mathbb{Z}) & \longrightarrow & H^2(W_{an}, \mathbb{Z}) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \frac{H^1(W_{an}, \mathbb{C})}{F^1 H^1(W_{an}, \mathbb{C})} & \longrightarrow & \frac{H^2(Z_{an}, W_{an}, \mathbb{C})}{F^1 H^2(Z_{an}, W_{an}, \mathbb{C})} & \longrightarrow & \frac{H^2(Z_{an}, \mathbb{C})}{F^1 H^2(Z_{an}, \mathbb{C})} & \longrightarrow & \frac{H^2(W_{an}, \mathbb{C})}{F^1 H^2(W_{an}, \mathbb{C})} & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

The above diagram comes from the following 9-diagram in the category of sheaves (where the bottom row defines $\mathcal{O}_{Z_{an}}(-W_{an})^*$, and $j : Z_{an} - W_{an} \hookrightarrow Z_{an}$ is the inclusion).

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & j! \mathbb{Z}_{Z_{an}-W_{an}} & \longrightarrow & \mathbb{Z}_{Z_{an}} & \longrightarrow & \mathbb{Z}_{W_{an}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{Z_{an}}(-W_{an}) & \longrightarrow & \mathcal{O}_{Z_{an}} & \longrightarrow & \mathcal{O}_{W_{an}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{Z_{an}}(-W_{an})^* & \longrightarrow & \mathcal{O}_{Z_{an}}^* & \longrightarrow & \mathcal{O}_{W_{an}}^* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Let $\eta \in H^2(Z_{an}, \mathbb{Z})$ such that $\eta \mapsto 0$ both in $H^2(W_{an}, \mathbb{Z})$ and $\frac{H^2(Z_{an}, \mathbb{C})}{F^1 H^2(Z_{an}, \mathbb{C})}$ (i.e., $\eta \in NS(Z)$). By a diagram chase as before we get elements δ_1 and δ_2 in the group $\frac{H^1(W_{an}, \mathbb{C})}{F^1 H^1(W_{an}, \mathbb{C})}$. Both δ_1 and δ_2 are well-defined in the quotient group $\frac{H^1(W_{an}, \mathbb{C})}{F^1 H^1(W_{an}, \mathbb{C})} / \left(\text{Im}(H^1(W_{an}, \mathbb{Z})) + \text{Im} \left(\frac{H^1(Z_{an}, \mathbb{C})}{F^1 H^1(Z_{an}, \mathbb{C})} \right) \right)$.

Now note that

$$\frac{H^1(W_{an}, \mathbb{C})}{F^1 H^1(W_{an}, \mathbb{C})} / \left(\text{Im}(H^1(W_{an}, \mathbb{Z}) + \text{Im} \left(\frac{H^1(Z_{an}, \mathbb{C})}{F^1 H^1(Z_{an}, \mathbb{C})} \right)) \right) \cong J \left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} \right).$$

Hence we get 2 maps $\eta \mapsto \bar{\delta}_1$ and $\eta \mapsto \bar{\delta}_2$ from

$$\mathbb{Z} \cong \mathbb{Z}[\eta] \rightarrow J \left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} \right)$$

where $\bar{\delta}$ denotes the class of δ in the quotient $J \left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} \right)$.

We claim that these two maps are nothing but our previously defined maps ϕ'' and ψ'' respectively. It is clear that the map $\eta \mapsto \bar{\delta}_2$ is equal to $\phi''(\eta)$. This is because we got $\bar{\delta}_2$ by first taking a lift of η , say β_2 in $\text{Pic}(Z_{an})$, then restricting to $\text{Pic}(W_{an})$ to get γ_2 and finally taking the class $\bar{\delta}_2 \in J \left(\frac{H^1(W_{an}, \mathbb{Z})}{H^1(Z_{an}, \mathbb{Z})} \right)$. This is exactly how $\phi''(\eta)$ was defined, so $\bar{\delta}_2 = \phi''(\eta)$. It is also clear that $\bar{\delta}_1 = \psi''(\eta)$ as the extension class map is defined exactly the same way as the map $\eta \mapsto \bar{\delta}_1$. Now by lemma 2.8, we have that $\bar{\delta}_1 = \bar{\delta}_2$. This implies that $\phi'' = \psi''$. \square

Remark 4.5. In a similar vein, using lemma 2.8, one can show that the cycle class map with values in Deligne-Beilinson cohomology restricts to the Abel-Jacobi mapping, on cycles which are homologically trivial. This is essentially the argument given in [EV], though the need to appeal to lemma 2.8 is not brought out explicitly there.

Let $Z = \tilde{Y}$ and $W = \tilde{F}$ in lemma 4.4. Let $\eta \in \text{Im}(H^1(X, \mathcal{H}_X^1) \rightarrow H^2(Y_{an}, \mathbb{Z}))$, and let $\eta \mapsto \tilde{\eta} \in H^2(\tilde{Y}_{an}, \mathbb{Z})$. Then, $\tilde{\eta}$ satisfies the conditions of lemma 4.4, i.e., $\tilde{\eta} \in NS(\tilde{Y})$ and $\tilde{\eta} \mapsto 0 \in H^2(\tilde{F}_{an}, \mathbb{Z})$. Clearly,

$$\psi'(\eta) = \psi''(\tilde{\eta}),$$

and

$$\phi'(\eta) = \phi''(\tilde{\eta}).$$

Hence,

$$\psi' = \phi'$$

which proves the commutativity of diagram (*). This finishes the proof of our main result, Theorem 1.1.

5. AN EXAMPLE

In this section we give an example of an integral projective variety X over \mathbb{C} which is not normal, for which we have a strict inclusion

$$NS(X) \subsetneq \{ \alpha \in H^2(X, \mathbb{Z}) \mid \alpha \text{ is Zariski locally trivial and } \alpha_{\mathbb{C}} \in F^1 H^2(X, \mathbb{Z}) \}.$$

Our variety will have the property that its normalization Y is non-singular, and the normalization map $\pi : Y \rightarrow X$ is bijective. Then $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ as mixed Hodge structures, and the subspaces of Zariski locally trivial classes

correspond. Hence the desired property of X is equivalent to the strictness of the first inclusion

$$NS(X) \subsetneq NS(Y) \subset H^2(Y, \mathbb{Z}).$$

We will make use of a variant of a construction in [Ha], III, Ex. 5.9 (see also [Ha], II, Ex. 5.16b). If V is a non-singular variety over \mathbb{C} , then following [Ha], an *infinitesimal extension* of V by an invertible \mathcal{O}_V -module \mathcal{L} is a scheme W with $W_{red} = V$, such that the nilradical \mathcal{I} of \mathcal{O}_W has square zero (so that it is an \mathcal{O}_V -module), and there is an \mathcal{O}_V -isomorphism $\mathcal{I} \cong \mathcal{L}$. In other words, there is an exact sequence of sheaves of \mathcal{O}_W -modules

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_V \rightarrow 0.$$

There is a corresponding exact sequence of sheaves

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{O}_W^* \rightarrow \mathcal{O}_V^* \rightarrow 0,$$

where \mathcal{L} is identified with the (multiplicative) subsheaf of units on W which restrict to 1 on V (the identification is given on sections by $s \mapsto 1 + s$). The latter exact sheaf sequence gives rise to an exact sequence on cohomology

$$H^1(V, \mathcal{L}) \rightarrow \text{Pic}W \rightarrow \text{Pic}V \xrightarrow{\delta} H^2(V, \mathcal{L}).$$

The following is an elaboration of [Ha], III, Ex. 5.9 (the proof is left as an exercise!).

Lemma 5.1. (i) *There is a natural bijection between isomorphism classes of infinitesimal extensions of V by \mathcal{L} and elements*

$$\alpha \in H^1(V, \mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \mathcal{L})).$$

(ii) *If W is the infinitesimal extension corresponding to α , the boundary map $\delta = \delta_\alpha : \text{Pic}V \rightarrow H^2(V, \mathcal{L})$ is expressible as the composition*

$$\text{Pic}V = H^1(V, \mathcal{O}_V^*) \xrightarrow{\text{dlog}} H^1(V, \Omega_{V/\mathbb{C}}^1) \xrightarrow{\cup \alpha} H^2(V, \mathcal{L}).$$

(iii) *Let $f : \mathcal{L} \rightarrow \mathcal{M}$ be a morphism of \mathcal{O}_X -modules, and $\alpha \mapsto f_*(\alpha)$ under the natural map*

$$f_* : H^1(V, \mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \mathcal{L})) \rightarrow H^1(V, \mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \mathcal{M})).$$

Let Z be the infinitesimal extension of V by \mathcal{M} determined by $f_(\alpha)$. Then there is a unique morphism of \mathbb{C} -schemes $\tilde{f} : Z \rightarrow W$, such that the corresponding morphism of reduced schemes is the identity on V , and such that there are commutative diagrams with exact rows*

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{L} & \rightarrow & \mathcal{O}_W & \rightarrow & \mathcal{O}_V \rightarrow 0 \\ & & f \downarrow & & \tilde{f}^* \downarrow & & \parallel \\ 0 & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{O}_Z & \rightarrow & \mathcal{O}_V \rightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc}
H^1(V, \mathcal{L}) & \rightarrow & \text{Pic}W & \rightarrow & \text{Pic}V & \xrightarrow{\delta_\alpha} & H^2(V, \mathcal{L}) \\
f_* \downarrow & & \tilde{f}^* \downarrow & & \parallel & & \downarrow f_* \\
H^1(V, \mathcal{M}) & \rightarrow & \text{Pic}Z & \rightarrow & \text{Pic}V & \xrightarrow{\delta_{f_*(\alpha)}} & H^2(V, \mathcal{M})
\end{array}$$

Example 5.2. In the above lemma, take $V = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, and $\mathcal{L} = \omega_V$, $\mathcal{M} = \mathcal{O}_V$, $f : \mathcal{L} \hookrightarrow \mathcal{M}$ any non-zero map (since $\omega_V \cong \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$, such maps f exist). Infinitesimal extensions of V by \mathcal{L} are classified by elements of

$$H^1(V, \mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \omega_V)) \cong H^1(V, \Omega_{V/\mathbb{C}}^1),$$

where we have identified $\mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \omega_V)$ with $\Omega_{V/\mathbb{C}}^1$ using the non-degenerate bilinear form

$$\Omega_{V/\mathbb{C}}^1 \otimes_{\mathcal{O}_V} \Omega_{V/\mathbb{C}}^1 \rightarrow \omega_V$$

given by the exterior product of 1-forms. Thus if $\alpha \in H^1(V, \Omega_{V/\mathbb{C}}^1)$, the corresponding cup-product map

$$H^1(V, \Omega_{V/\mathbb{C}}^1) \xrightarrow{\cup \alpha} H^2(V, \omega_V)$$

is just the product with α in the commutative graded ring

$$\oplus_{n \geq 0} H^n(V, \Omega_{V/\mathbb{C}}^n).$$

Thus if α is the cohomology class of a divisor D on V , then for any divisor E on V , we see that by (ii) of the lemma,

$$\delta_\alpha(E) = (D \cdot E) \in \mathbb{C} = H^2(V, \omega_V)$$

is the intersection product of D and E on the non-singular projective surface V .

We will choose α to be the cohomology class of $D = L_1 - L_2$, where $L_1 = \mathbb{P}_{\mathbb{C}}^1 \times \{0\}$ and $L_2 = \{0\} \times \mathbb{P}_{\mathbb{C}}^1$ are elements of the two rulings on $V = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$. Since $D^2 = -2$, $\alpha \neq 0$ and $W = (V, \mathcal{O}(\alpha))$ is a non-trivial infinitesimal extension of V by ω_V . Note that $H^1(V, \omega_V) = 0$, so that there is an exact sequence

$$0 \rightarrow \text{Pic}W \rightarrow \text{Pic}V \xrightarrow{(D \cdot)} \mathbb{Z} \rightarrow 0$$

(($D \cdot L_2$) = 1, so the map to \mathbb{Z} is surjective). Here $\text{Pic}V = \text{Pic}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$ is free abelian of rank 2, and as usual, we denote a representative of the class of $a[L_1] + b[L_2]$ by $\mathcal{O}_V(a, b)$.

Next, note that

$$f_*(\alpha) \in H^1(V, \mathcal{H}om_V(\Omega_{V/\mathbb{C}}^1, \mathcal{O}_V)) = 0,$$

since $\Omega_{V/\mathbb{C}}^1 \cong \mathcal{O}_V(-2, 0) \oplus \mathcal{O}_V(0, -2)$. Hence the infinitesimal extension Z of V by \mathcal{O}_V determined by $f_*(\alpha)$ is the trivial extension $(V, \mathcal{O}_V[\epsilon])$, where $\mathcal{O}_V[\epsilon]$ is the sheaf of dual numbers over \mathcal{O}_V .

There is an obvious way in which we may regard $Z = (V, \mathcal{O}_V[\epsilon])$ as a closed subscheme of $Y = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 = V \times \mathbb{P}_{\mathbb{C}}^1$, whose underlying reduced scheme is $V \times \{0\}$.

Finally, we define X to be the \mathbb{C} -scheme which is the pushout of Y and W along the morphisms $\tilde{f} : Z \rightarrow W$ and the above inclusion $i : Z \hookrightarrow Y$, so that there is a commutative pushout diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & Y = V \times \mathbb{P}_{\mathbb{C}}^1 \\ \tilde{f} \downarrow & & \downarrow \pi \\ W & \xrightarrow{j} & X \end{array}$$

Since \tilde{f} is a finite and bijective morphism, we see that for each affine open $U = \text{Spec } A$ in Y , $U \cap Z = \text{Spec } A/I$ is affine, and finite over the affine open subscheme $\tilde{f}(U \cap Z) = \text{Spec } B \subset W$. The image of U in X is then defined as the affine scheme $\text{Spec } C$, where C is the inverse image of B in A under the surjection $A \twoheadrightarrow A/I$. One shows easily that C is in fact a finitely generated \mathbb{C} -subalgebra of A , and A is a finite C -module with conductor ideal I . Further, the construction of C localizes well. Hence the local schemes $\text{Spec } C$ can be glued together to yield the \mathbb{C} -scheme X .

We claim that this scheme X has the desired properties, *i.e.*, X is an integral projective scheme over \mathbb{C} with normalization $\pi : Y \rightarrow X$, such that Y is non-singular and bijective with X , while $NS(X) \rightarrow NS(Y)$ is a strict inclusion.

That X is integral with Y as its normalization is clear, from the description of its affine open sets above. Next, since $\alpha = [D]$, and $(D \cdot (L_1 + L_2)) = 0$, there is a unique $\mathcal{H} \in \text{Pic} W$ such that $\mathcal{H} \otimes \mathcal{O}_V = \mathcal{O}_V(1, 1)$. Also $\text{Pic} Z \rightarrow \text{Pic} V$ is an isomorphism. We have $\text{Pic} Y = \mathbb{Z}^{\oplus 3}$, where we may regard the restriction map $\text{Pic} Y \rightarrow \text{Pic} V \times \{0\} = \text{Pic} V = \mathbb{Z}^{\oplus 2}$ as projection on the first 2 factors. Hence we see that the very ample invertible sheaf $\mathcal{O}_Y(1, 1, 1)$ has the property that there is an isomorphism $\tilde{f}^* \mathcal{H} \cong \mathcal{O}_Y(1, 1, 1) \otimes \mathcal{O}_Z$. From the Mayer-Vietoris sequence of sheaves of rings

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y \oplus j_* \mathcal{O}_W \rightarrow (i \circ \pi)_* \mathcal{O}_Z \rightarrow 0$$

we have a corresponding sequence of sheaves of unit groups

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \pi_* \mathcal{O}_Y^* \oplus j_* \mathcal{O}_W^* \rightarrow (i \circ \pi)_* \mathcal{O}_Z^* \rightarrow 0$$

leading to an exact sequence

$$H^0(Y, \mathcal{O}_Y^*) \oplus H^0(W, \mathcal{O}_W^*) \rightarrow H^0(Z, \mathcal{O}_Z^*) \rightarrow \text{Pic} X \rightarrow \text{Pic} Y \oplus \text{Pic} W \rightarrow \text{Pic} Z.$$

Hence there exists an invertible sheaf \mathcal{A} on X with $\pi^* \mathcal{A} = \mathcal{O}_Y(1, 1, 1)$ (and $j^* \mathcal{A} = \mathcal{H}$). Since π is finite, and $\pi^* \mathcal{A}$ is ample on Y , we have that \mathcal{A} is ample on X . Hence X is projective.

From the exact sequence

$$0 \rightarrow \omega_V \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_V \rightarrow 0,$$

we see that $H^0(W, \mathcal{O}_W) \rightarrow H^0(V, \mathcal{O}_V) = \mathbb{C}$ is an isomorphism. On the other hand, we see at once that $H^0(Z, \mathcal{O}_Z) = \mathbb{C}[\epsilon]$ is the ring of dual numbers. Hence we get analogous formulas for the unit groups. Thus there is an exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \text{Pic} X \rightarrow \text{Pic} Y \oplus \text{Pic} W \rightarrow \text{Pic} Z \rightarrow 0$$

(note that $\text{Pic}Z = \text{Pic}V$ is a quotient of $\text{Pic}Y$). Since $\text{Pic}W \hookrightarrow \text{Pic}Z$ is an inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}^{\oplus 2}$ as a direct summand, while $\text{Pic}Y \rightarrow \text{Pic}Z$ is the projection $\mathbb{Z}^{\oplus 3} \twoheadrightarrow \mathbb{Z}^{\oplus 2}$, we see that $\text{image}(\text{Pic}X \rightarrow \text{Pic}Y = NS(Y))$ is a direct summand $\mathbb{Z}^{\oplus 2} \hookrightarrow \mathbb{Z}^{\oplus 3}$. Thus $NS(X) = \mathbb{Z}^{\oplus 2}$ is strictly contained in $NS(Y) = \mathbb{Z}^{\oplus 3}$. \square

REFERENCES

- [BS1] L. Barbieri-Viale and V. Srinivas, *The Néron-Severi group and the mixed Hodge structure on H^2* , J. reine angew. Math. 450 (1994), 37-42.
- [BS2] L. Barbieri-Viale and V. Srinivas, *On the Néron-Severi group of a singular variety*, J. reine angew. Math. 435 (1993), 65-82.
- [C] J. Carlson, *The extension class of a mixed Hodge structure*, in *Algebraic Geometry, Bowdoin 1985*, Proc. Sym. Pure Math. Vol. 46, Part 2, Amer. Math. Soc. (1987).
- [Co] A. Collino, *Washnitzer's conjecture and the cohomology of a variety with a single isolated singularity*, Ill. J. Math. 29 (1985) 353-364.
- [D] P. Deligne, *Théorie de Hodge, II, III*, Publ. Math. IHES (1971) 5-57 and (1974) 5-77.
- [DB] Ph. Du Bois, *Complexe de De Rham filtré d'une variété singulière*, Bull. Math. Soc. France 109 (1981), 41-81.
- [EV] H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, in *Beilinson's Conjectures on special values of L-functions*, Perspectives in Math. 4, Academic Press, New York 1988.
- [GH] P.A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, (1978), New York.
- [H] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. No. 52, Springer-Verlag, (1977).
- [J] U. Jannsen, *Mixed motives and algebraic K-theory*, Lect. Notes Math. 1400, Springer-Verlag, Berlin (1990).
- [L] S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, (1950).
- [PS] R. Parimala and V. Srinivas, *Analogues of the Brauer group for algebras with involution*, Duke Math. J. 66 (1992), 207-237.
- [V] J.-L. Verdier, *Classe d'homologie associée à un cycle*, in: Séminaire de Géométrie Analytique, Astérisque, 36-37 (1976), 101-151.

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